Mathematics 2019
Delhi Set-1

General Instructions:

(i) **All** questions are compulsory.

(ii) This question paper contains **29** questions divided into four sections A, B, C and D. Section **A** comprises of 4 questions of **one mark** each, Section B comprises of 8 questions of **two marks** each, Section C comprises of 11 questions of **four marks** each and Section D comprises of 6 questions of **six marks** each.

(iii) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.

(iv) There is no overall choice. However, internal choice has been provided in 1 question of Section A, 3 questions of Section B, 3 questions of Section C and 3 questions of Section D. You have to attempt only **one** of the alternatives in all such questions.

(v) Use of calculators is not permitted. You may ask logarithmic tables, if required.

Question 1

If A and B are square matrices of the same order 3, such that |A| = 2 and AB = 2I, write the value of |B|.

SOLUTION:

We have, AB = 2I

$$|AB| = |2I|$$
$$\Rightarrow |A|\,|B| = 8$$
$$\Rightarrow 2\,|B| = 8 \qquad (\text{Given } |A| = 2)$$
$$\Rightarrow |B| = 4$$

Question 2

If $f(x) = x + 1$, find $\frac{d}{dx}(fof)\ (x)$.

SOLUTION:

Given: $f(x) = x + 1$

$$fof(x) = (x+1) + 1 = x + 2$$
$$\frac{d}{dx}(fof)(x) = \frac{d}{dx}(x+2) = 1$$

Question 3

Find the order and the degree of the differential equation $x^2 \frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^4$.

SOLUTION:

The highest order derivative present in the given differential equations is $\frac{d^2y}{dx^2}$, so its order is 2. It is a polynomial $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ and the highest power raised to $\frac{d^2y}{dx^2}$ is 1, so its degree is 1.

Question 4

If a line makes angles 90°, 135°, 45° with the *x*, *y* and *z* axes respectively, find its direction cosines.

OR

Find the vector equation of the line which passes through the point (3, 4, 5) and is parallel to the vector $2\hat{i} + 2\hat{j} - 3\hat{k}$.

SOLUTION:

A line makes $90^\circ,\ 135^\circ,\ 45^\circ$ with $x,\ y$ and z axes respectively.
Therefore, Direction cosines of the line are $\cos 90^\circ,\ \cos 135^\circ$ and $\cos 45^\circ$
$\Rightarrow$ Direction cosines of the line are $0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

OR

Vector equation of a line which passes through a point $(3, 4, 5)$ and parallel to the vector $2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\widehat{\mathbf{k}}$ is

$$\vec{r} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 5\widehat{\mathbf{k}} + \mu\left(2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\widehat{\mathbf{k}}\right)$$

Question 5

Examine whether the operation *defined on *R* by $a * b = ab + 1$ is (i) a binary or not. (ii) if a binary operation, is it associative or not ?

SOLUTION:

The given operation is $a * b = ab + 1$

If any operation is a binary operation then it must follow the closure property.

Let $a \in R$, $b \in R$

then $a * b \in R$

also $ab + 1 \in R$

i.e. $a * b \in R$

so $*$ on R satisfies the closure property

Now if this binary operation satisfies associative law then

$(a * b) * c = a * (b * c)$

$$(a * b) * c = (ab + 1) * c$$
$$= (ab + 1)c + 1$$
$$= abc + c + 1$$

$$a * (b * c) = a * (bc + 1)$$
$$= a(bc + 1) + 1$$
$$= abc + a + 1$$

$\therefore (a * b) * c \neq a * (b * c)$

i.e., $*$ operation does not follow associative law.

Question 6

Find a matrix A such that $2A - 3B + 5C = O$, where

$$B = \begin{bmatrix} -2 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 0 & -2 \\ 7 & 1 & 6 \end{bmatrix}.$$

SOLUTION:

Given: $2A - 3B + 5C = 0$

$$\Rightarrow 2A - 3\begin{bmatrix} -2 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix} + 5\begin{bmatrix} 2 & 0 & -2 \\ 7 & 1 & 6 \end{bmatrix} 0$$

$$\Rightarrow 2A - \begin{bmatrix} -6 & 6 & 0 \\ 9 & 3 & 12 \end{bmatrix} + \begin{bmatrix} 10 & 0 & -10 \\ 35 & 5 & 30 \end{bmatrix} = 0$$

$$\Rightarrow 2A + \begin{bmatrix} 10 + 6 & 0 - 6 & -10 - 0 \\ 35 - 9 & 5 - 3 & 30 - 12 \end{bmatrix} = 0$$

$$\Rightarrow 2A = -\begin{bmatrix} 16 & -6 & -10 \\ 26 & 2 & 18 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} -8 & 3 & 5 \\ -13 & -1 & -9 \end{bmatrix}$$

Question 7

Find : $\int \frac{\sec^2 x}{\sqrt{\tan^2 x+4}} dx.$

SOLUTION:

$I = \int \frac{\sec^2 x}{\sqrt{\tan^2 x+4}} dx$

Let $\tan x = t$

$\sec^2 x dx = dt$

So, $I = \int \frac{dt}{\sqrt{t^2+4}}$

or, $I = \int \frac{dt}{\sqrt{t^2+2^2}}$

Since, we know

$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln|x + \sqrt{a^2+x^2}| + C$

$I = \ln|t + \sqrt{t^2+4}| + C$

$i.e$

$I = \ln\left|\tan x + \sqrt{\tan^2 x + 4}\right| + C$

Question 8

Find : $\int \sqrt{1 - \sin 2x}\, dx,\ \frac{\pi}{4} < x < \frac{\pi}{2}$

OR

Find : $\int \sin^{-1}(2x)\, dx.$

SOLUTION:

$I = \int \sqrt{1 - \sin 2x}\, dx$

$I = \int \sqrt{\sin^2 x + \cos^2 x - 2 \sin x \cos x}\, dx$

$I = \int (\sin x - \cos x) dx$

$I = \int \sin x\, dx - \int \cos x\, dx$

$I = -\cos x - \sin x + C$

OR

$\int \sin^{-1}(2x)\, dx$

Using ILATE rule

$x\sin^{-1}2x - \int \frac{2x}{\sqrt{1-4x^2}}dx$

$x\sin^{-1}2x + \frac{1}{4}\int \frac{-8x}{\sqrt{1-4x^2}}dx$

Taking $1 - 4x^2 = t$

$\Rightarrow -8x\, dx = dt$

$x\sin^{-1}2x + \frac{1}{4}\int \frac{dt}{\sqrt{t}}$

$x\sin^{-1}2x + \frac{1}{4}\frac{2t^{1/2}}{1} + C$

$= x\sin^{-1}2x + \frac{t^{1/2}}{2} + C$

$= x\sin^{-1}2x + \frac{\sqrt{1-4x^2}}{2} + C$

Question 9

Form the differential equation representing the family of curves $y = e^{2x}(a + bx)$, where 'a' and 'b' are arbitrary constants.

SOLUTION:

Given: $y = e^{2x}(a+bx)$

Differentiating the above equation, we get

$\frac{dy}{dx} = be^{2x} + 2(a+bx)e^{2x}$

$= \frac{dy}{dx} = be^{2x} + 2y \quad \ldots(\text{i}) \quad [\because y = e^{2x}(a+bx)]$

differentiating the above equation, we get

$\frac{d^2y}{dx^2} = 2\,be^{2x} + 2\frac{dy}{dx}$

$= \frac{d^2y}{dx^2} = 2\left(\frac{dy}{dx} - 2y\right) + 2\frac{dy}{dx} \quad \left[\because \text{ from } (\text{i}) \text{ we get, } be^{2x} = \frac{dy}{dx} - 2y\right]$

$= \frac{d^2y}{dx^2} = 4\frac{dy}{dx} - 4y$

Hence, the required differential equation is $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$

Question 10

If the sum of two unit vectors is a unit vector, prove that the magnitude of their difference is $\sqrt{3}$.

OR

If

$\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k},\ \vec{b} = \hat{i} - 2\hat{j} + \hat{k}$ and $\vec{c} = -3\hat{i} + \hat{j} + 2\hat{k}$, find $\left[\vec{a}\,\vec{b}\,\vec{c}\right]$.

SOLUTION:

Let three unit vectors are *a, b* and *c*
given that the sum of the unit vectors is a unit vector.

$\therefore a + b = c$
or $|c|^2 = |a + b|^2$
or $|c|^2 = |a|^2 + |b|^2 + 2|a||b|\cos\theta$

or $1 = 1 + 1 + 2\cos\theta$ [$\because |a| = |b| = |c| = 1$ (unit vector)]

$\Rightarrow \cos\theta = -1/2$ ...(1)

Now, $|a - b|^2 = |a|^2 + |b|^2 - 2|a||b|\cos\theta$
$|a - b|^2 = [1+1+1]$
$|a - b| = \sqrt{3}$

OR

$$\left[\vec{a}\,\vec{b}\,\vec{c}\right] = \begin{vmatrix} 2 & 3 & 1 \\ 1 & -2 & 1 \\ -3 & 1 & 2 \end{vmatrix} = 2(-4-1) - 3(2+3) + 1(1-6)$$
$$= -30$$

Question 11

A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let *A* be the event "number is even" and *B* be the event "number is marked red". Find whether the events *A* and *B* are independent or not.

Solution Not Available

Question 12

A die is thrown 6 times. If "getting an odd number" is a "success", what is the probability of (i) 5 successes? (ii) atmost 5 successes?

OR

The random variable X has a probability distribution P(X) of the following form, where 'k is some number.

$$P(X = x) = \begin{cases} k, & \text{if } x = 0 \\ 2k, & \text{if } x = 1 \\ 3k, & \text{if } x = 2 \\ 0, & \text{otherwise} \end{cases}$$

Determine the value of 'k.

Question 13

Show that the relation R on $\mathbb{R}$ defined as R = {(a, b) : $a \le b$}, is reflexive, and transitive but not symmetric.

OR

Prove that the function f: N → N, defined by $f(x) = x^2 + x + 1$ is one-one but not onto. Find inverse of f: N → S, where S is range of f.

SOLUTION:

R = {(a, b); a ≤ b}
Clearly (a, a) ∈ R as a = a.
∴ R is reflexive.
Now,
(2, 4) ∈ R (as 2 < 4)
But, (4, 2) ∉ R as 4 is greater than 2.
∴ R is not symmetric.
Now, let (a, b), (b, c) ∈ R.
Then,
a ≤ b and b ≤ c
⇒ a ≤ c
⇒ (a, c) ∈ R
∴ R is transitive.
Hence, R is reflexive and transitive but not symmetric.

OR

The given function is

$f: N \rightarrow N$

$f(x) = x^2 + x + 1$

Let x_1, x_2 6N

$$\text{So let } f(x_1) = f(x_2)$$

$$x_1^2 + x_1 + 1 = x_2^2 + x_2 + 1$$

$$x_1^2 - x_2^2 + x_1 - x_2 = 0$$

$$(x_1 - x_2)(x_1 + x_2 + 1) = 0$$

$$\because x_2 = x_1$$

$$\text{or } x_2 = -x_1 - 1$$

$$\because x_1 \in N$$

$$\therefore -x_1 - 1 \in N$$

$$\text{So } x_2 \neq -x_1 - 1$$

$$\because f(x_2) = f(x_1) \text{ only for } x_1 = x_2$$

So $f(x)$ is one-one function.

$$\because f(x) = x^2 + x + 1$$

$$f(x) = \left(x + \tfrac{1}{2}\right)^2 + \tfrac{3}{4}$$

Which is an increasing function.

$f(1) = 3$

$\because$ Range of $f(x)$ will be {3, 7, ...}

Which is a subset of N.

So it is an into function.

i.e., $f(x)$ is not an onto function.

$$\text{let } y = x^2 + x + 1$$

$$x^2 + x + 1 - y = 0$$

$$x = \frac{-1 \pm \sqrt{1 - 4(1 - y)}}{2}$$

$$x = \frac{-1 \pm \sqrt{4y - 3}}{2}$$

So two possibilities are their for $f^{-1}(x)$

$f^{-1}(x) = \frac{-1+\sqrt{4x-3}}{2}, \frac{-1-\sqrt{4x-3}}{2}$ and we know $f^{-1}(3) = 1$ because $f(1) = 3$

$$\text{so } f^{-1}(x) = \frac{-1+\sqrt{4x-3}}{2}$$

Question 14

Solve: $\tan^{-1} 4x + \tan^{-1} 6x = \frac{\pi}{4}$.

SOLUTION:

We have $\tan^{-1} 4x + \tan^{-1} 6x = \frac{\pi}{4}$

$\Rightarrow \quad \tan\left(\tan^{-1} 4x + \tan^{-1} 6x\right) = \tan\frac{\pi}{4}$

$\Rightarrow \quad \frac{\tan(\tan^{-1} 4x) + \tan(\tan^{-1} 6x)}{1 - \tan(\tan^{-1} 4x)\cdot\tan(\tan^{-1} 6x)} = 1$

$\Rightarrow \quad \frac{4x+6x}{1-4x\cdot 6x} = 1$

$\Rightarrow \quad \frac{10x}{1-24x^2} = 1$

$\Rightarrow \quad 24x^2 + 10x - 1 = 0$

$\Rightarrow \quad 24x^2 + 12x - 2x - 1 = 0$

$\Rightarrow \quad 12x(2x+1) - 1(2x+1) = 0$

$\Rightarrow \quad (2x+1)(12x-1) = 0$

$\Rightarrow \quad x = -\frac{1}{2}, \frac{1}{12}$

But $x = \frac{-1}{2}$ does not satisfy the equation as the LHS will become negative

Therefore, the value of x is $\frac{1}{12}$.

Question 15

Using properties of determinants, prove that $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$.

SOLUTION:

$$\mathrm{L.H.S} = \begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$$

$R_2 \longrightarrow R_2 - R_1,\ R_3 \longrightarrow R_3 - R_1$

$$= \begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 1-a^2 & -a+1 & 0 \\ 3-a^2-2a & 3-2a-1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 1-a^2 & 1-a & 0 \\ 3-a^2-2a & 2-2a & 0 \end{vmatrix}$$

Expanding along C_3

$= 1\left[\left(1-a^2\right)(2-2a) - (1-a)\left(3-a^2-2a\right)\right]$

$= 2(1-a)(1-a)(1+a) - (1-a)\left(3-a^2-2a\right)$

$= (1-a)\left[2\left(1-a^2\right) - 3 + a^2 + 2a\right]$

$= (1-a)\left(2a-a^2-1\right)$

$= (a-1)^3$

= RHS

Question 16

If $\log\left(x^2 + y^2\right) = 2\tan^{-1}\left(\frac{y}{x}\right)$, show that $\frac{dy}{dx} = \frac{x+y}{x-y}$.

OR

If $x^y - y^x = a^b$, find $\frac{dy}{dx}$.

SOLUTION:

$\text{We have, } \log\left(x^2+y^2\right) = 2\tan^{-1}\left(\frac{y}{x}\right)$

$\Rightarrow \frac{1}{2}\log\left(x^2+y^2\right) = \tan^{-1}\left(\frac{y}{x}\right)$

Differentiate with respect to x, we get,

$\Rightarrow \frac{1}{2}\frac{d}{dx}\log\left(x^2+y^2\right) = \frac{d}{dx}\tan^{-1}\left(\frac{y}{x}\right)$

$$\Rightarrow \frac{1}{2}\left(\frac{1}{x^2+y^2}\right)\frac{d}{dx}\left(x^2+y^2\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{d}{dx}\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{1}{2}\left(\frac{1}{x^2+y^2}\right)\left[2x + 2y\frac{dy}{dx}\right] = \frac{x^2}{\left(x^2+y^2\right)}\left[\frac{x\frac{dy}{dx} - y\frac{d}{dx}(x)}{x^2}\right]$$

$$\Rightarrow \left(\frac{1}{x^2+y^2}\right)\left(x+y\frac{dy}{dx}\right)=\frac{x^2}{(x^2+y^2)}\left[\frac{x\frac{dy}{dx}-y\frac{d}{dx}(x)}{x^2}\right]$$

$$\Rightarrow \left(\frac{1}{x^2+y^2}\right)\left(x+y\frac{dy}{dx}\right)=\frac{x^2}{(x^2+y^2)}\left[\frac{x\frac{dy}{dx}-y(1)}{x^2}\right]$$

$$\Rightarrow x+y\frac{dy}{dx}=x\frac{dy}{dx}-y$$

$$\Rightarrow y\frac{dy}{dx}-x\frac{dy}{dx}=-y-x$$

$$\Rightarrow \frac{dy}{dx}(y-x)=-(y+x)$$

$$\Rightarrow \frac{dy}{dx}=\frac{-(y+x)}{y-x}$$

$$\Rightarrow \frac{dy}{dx}=\frac{x+y}{x-y}$$

OR

The given function is $x^y - y^x = a^b$

Let $x^y = u$ and $y^x = v$

Then, the function becomes $u - v = a^b$

$$\frac{du}{dx}-\frac{dv}{dx}=0 \qquad \ldots\ldots(1)$$

$$u = x^y$$
$$\Rightarrow \log u = \log\left(x^y\right)$$
$$\Rightarrow \log u = y\log x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx}=\log x\frac{dy}{dx}+y\cdot\frac{d}{dx}(\log x)$$
$$\Rightarrow \frac{du}{dx}=u\left[\log x\frac{dy}{dx}+y\cdot\frac{1}{x}\right]$$
$$\Rightarrow \frac{du}{dx}=x^y\left(\log x\frac{dy}{dx}+\frac{y}{x}\right) \qquad \ldots(2)$$

$$v = y^x$$
$$\Rightarrow \log v = \log\left(y^x\right)$$
$$\Rightarrow \log v = x \log y$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{v} \cdot \frac{dv}{dx} = \log y \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log y)$$
$$\Rightarrow \frac{dv}{dx} = v\left(\log y \cdot 1 + x \cdot \frac{1}{y} \cdot \frac{dy}{dx}\right)$$
$$\Rightarrow \frac{dv}{dx} = y^x\left(\log y + \frac{x}{y}\frac{dy}{dx}\right) \qquad \ldots(3)$$

From (1), (2), and (3), we obtain

$$x^y\left(\log x \frac{dy}{dx} + \frac{y}{x}\right) - y^x\left(\log y + \frac{x}{y}\frac{dy}{dx}\right) = 0$$
$$\Rightarrow x^y \log x \frac{dy}{dx} - xy^{x-1}\frac{dy}{dx} + x^{y-1}y - y^x \log y = 0$$
$$\Rightarrow \left(x^y \log x - xy^{x-1}\right)\frac{dy}{dx} = y^x \log y - x^{y-1}y$$
$$\Rightarrow \frac{dy}{dx} = \frac{y^x \log y - x^{y-1}y}{\left(x^y \log x - xy^{x-1}\right)}$$

Question 17

If $y = (\sin^{-1}x)^2$, prove that $\left(1 - x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0$.

SOLUTION:

Here,

$y=\left(\sin^{-1}x\right)^2$

Now,

$y_1=2\ \sin^{-1}x\ \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow y_2=\frac{2}{1-x^2}+\frac{2x\ \sin^{-1}x}{(1-x^2)^{3/2}}$

$\Rightarrow y_2=\frac{2}{1-x^2}+\frac{2x\ \sin^{-1}x}{(1-x^2)\sqrt{1-x^2}}$

$\Rightarrow y_2=\frac{2}{1-x^2}+\frac{xy_1}{(1-x^2)}$

$\Rightarrow y_2\left(1-x^2\right)=2+xy_1$

$\Rightarrow y_2\left(1-x^2\right)-xy_1-2=0$

Therefore, $\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-2=0$

Hence proved.

Question 18

Find the equation of tangent to the curve $y=\sqrt{3x-2}$ which is parallel to the line $4x-2y+5=0$. Also, write the equation of normal to the curve at the point of contact.

SOLUTION:

Slope of the given line is 2

Let (x_1,y_1)be the point where the tangent is drawn to the curve $y=\sqrt{3x-2}$

Since, the point lies on the curve.

Hence, $y_1=\sqrt{3x_1-2}$ $\qquad\ldots(1)$

Now, $y=\sqrt{3x-2}$

$\Rightarrow\frac{dy}{dx}=\frac{3}{2\sqrt{3x-2}}$

Slope of tangent at $(x_1,y_1)=\frac{3}{2\sqrt{3x_1-2}}$

Given that

Slope of tangent $=$ slope of the given line

$\Rightarrow \frac{3}{2\sqrt{3x_1-2}} = 2$

$\Rightarrow 3 = 4\sqrt{3x_1 - 2}$

$\Rightarrow 9 = 16\left(3x_1 - 2\right)$

$\Rightarrow \frac{9}{16} = 3x_1 - 2$

$\Rightarrow 3x_1 = \frac{9}{16} + 2 = \frac{9+32}{16} = \frac{41}{16}$

$\Rightarrow x_1 = \frac{41}{48}$

Now, $y_1 = \sqrt{\frac{123}{48} - 2} = \sqrt{\frac{27}{48}} = \sqrt{\frac{9}{16}} = \frac{3}{4}$ [From (1)]

$\therefore (x_1, y_1) = \left(\frac{41}{48}, \frac{3}{4}\right)$

Equation of tangent is,

$y - y_1 = m\ (x - x_1)$

$\Rightarrow y - \frac{3}{4} = 2\ \left(x - \frac{41}{48}\right)$

$\Rightarrow \frac{4y-3}{4} = 2\left(\frac{48x-41}{48}\right)$

$\Rightarrow 24y - 18 = 48x - 41$

$\Rightarrow 48x - 24y - 23 = 0$

Equation of normal at the point of contact will be

$y - y_1 = \frac{-1}{m}\left(x - x_1\right)$

$\Rightarrow y - \frac{3}{4} = \frac{-1}{2}\left(x - \frac{41}{48}\right)$

$\Rightarrow \frac{4y-3}{4} = \frac{-1}{2}\left(x - \frac{41}{48}\right)$

$\Rightarrow \frac{4y-3}{2} = \left(\frac{41}{48} - x\right)$

$\Rightarrow \frac{4y-3}{2} = \frac{41-48x}{48}$

$\Rightarrow 4y - 3 = \frac{41-48x}{24}$

$96y - 72 = 41 - 48x$

$\Rightarrow 48x + 96y = 113$

Question 19

Find: $\int \frac{3x+5}{x^2+3x-18} dx.$

SOLUTION:

Let $I = \int \frac{(3x+5)\,dx}{x^2+3x-18}$

$I = \int \frac{(3x+5)\,dx}{(x+6)\,(x-3)}$

let $\frac{3x+5}{(x+6)\,(x-3)} = \frac{A}{x+6} + \frac{B}{x-3}$

so $3x+5 = A\,(x-3) + B\,(x+6)$

On comparing,

$A + B = 3$ .... (i)

$-3A + 6B = 5$ .... (ii)

$-3A + 6(3 - A) = 5$

$-3A + 18 - 6A = 5$

$A = \frac{-13}{-9} = \frac{13}{9}$ and $B = 3 - A = 3 - \frac{13}{9} = \frac{14}{9}$

So, $\int \frac{(3x+5)dx}{(x+6)(x-3)} = \int \frac{13\,dx}{9(x+6)} + \int \frac{14dx}{9(x-3)}$

$= \frac{13}{9} \ln\,(x+6) + \frac{14}{9} \ln\,(x-3) + \mathrm{C}$

Question 20

Prove that $\int_0^a f\,(x)\,\mathrm{d}\,x = \int_0^a f\,(a-x)\,\mathrm{d}\,x$, hence evaluate $\int_0^\pi \frac{x \sin x}{1+\cos^2 x}\,\mathrm{d}\,x.$

SOLUTION:

To prove: $\int_0^a f(x)\,\mathrm{d}x = \int_0^a f(a-x)\,\mathrm{d}x$

Proof: Let

$t = a - x$

$\Rightarrow dt = -dx$

When $x = 0,\ t = a$

When $x = a,\ t = 0$

Putting the value of x in LHS

$\int_a^0 f(a-t)(-\mathrm{d}t)$

$= -\int_a^0 f(a-t)(\mathrm{d}t)$

$= \int_0^a f(a-t)(\mathrm{d}t)$

$= \int_0^a f(a-x)(\mathrm{d}x) \qquad \left(\because \int_a^b f(t)\,\mathrm{d}t = \int_a^b f(x)\,\mathrm{d}x\right)$

$= \text{RHS}$

Using this we can solve the given question as follows:

$$I = \int_0^{\pi} f(x)\,dx = \int_0^{\pi} f(\pi - x)\,dx$$

$$\Rightarrow 2I = \int_0^{\pi} f(x)\,dx + \int_0^{\pi} f(\pi - x)\,dx = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx + \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}dx$$

$$\Rightarrow 2I = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx + \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2(\pi - x)}dx$$

$$\Rightarrow 2I = \int_0^{\pi} \frac{\pi\sin x}{1+\cos^2 x}dx$$

$Let,\ \cos x = t \Rightarrow -\sin x\,dx = dt$

$$\Rightarrow 2I = -\int_1^{-1} \frac{\pi}{1+t^2}dt = -\pi\left[\tan^{-1} t\right]_1^{-1} = -\pi\left(-\frac{\pi}{4} - \frac{\pi}{4}\right) = \frac{\pi^2}{2}$$

$$\therefore I = \frac{\pi^2}{4}$$

Question 21

Solve the differential equation: $x\,dy - y\,dx = \sqrt{x^2+y^2}dx$, given that $y = 0$ when $x = 1$.

OR

Solve the differential equation: $(1+x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$, subject to the initial condition $y(0) = 0$.

SOLUTION:

$$xdy - ydx = \sqrt{x^2+y^2}\,dx$$

$$\Rightarrow xdy = \left[y + \sqrt{x^2+y^2}\right]dx$$

$$\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x} \quad \ldots(1)$$

Let $F(x,y) = \frac{y+\sqrt{x^2+y^2}}{x}$.

$$\therefore\ F(\lambda x,\lambda y) = \frac{\lambda x+\sqrt{(\lambda x)^2+(\lambda y)^2}}{\lambda x} = \frac{y+\sqrt{x^2+y^2}}{x} = \lambda^0 \cdot F(x,y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

$y = vx$

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of v and $\frac{dy}{dx}$ in equation (1), we get:

$$v+x\frac{dv}{dx}=\frac{vx+\sqrt{x^2+(vx)^2}}{x}$$
$$\Rightarrow v+x\frac{dv}{dx}=v+\sqrt{1+v^2}$$
$$\Rightarrow \frac{dv}{\sqrt{1+v^2}}=\frac{dx}{x}$$

Integrating both sides, we get:

$$\log\left|v+\sqrt{1+v^2}\right|=\log|x|+\log C$$
$$\Rightarrow \log\left|\frac{y}{x}+\sqrt{1+\frac{y^2}{x^2}}\right|=\log|Cx|$$
$$\Rightarrow \log\left|\frac{y+\sqrt{x^2+y^2}}{x}\right|=\log|Cx|$$
$$\Rightarrow y+\sqrt{x^2+y^2}=Cx^2$$

This is the required solution of the given differential equation.

OR

The given differential equation can be written as:

$$\frac{dy}{dx}+\frac{2x}{1+x^2}y=\frac{4x^2}{1+x^2}\ldots(1)$$

This is a linear differential equation of the form $\frac{dy}{dx}+Py=Q$

$P=\frac{2x}{1+x^2}$ and $Q=\frac{4x}{1+x^2}$

$$\text{I.F}=e^{\int Pdx}=e^{\int\frac{2x}{1+x^2}dx}=e^{\log(1+x^2)}=1+x^2$$

Multipying both sides of (1) by $\text{I.F} = (1+x^2)$, we get

$(1+x^2)\frac{dy}{dx} + 2xy = 4x^2$

Integrating both sides with respect to x, we get

$y(1+x^2) = \int 4x^2 \text{dx} + C$

$y(1+x^2) = \frac{4x^3}{3} + \text{C}\ldots(2)$

Given $y = 0$, when $x = 0$

Substituting $\text{x} = 0$ and $\text{y} = 0$ in (1), we get

$0 = 0 + C \Rightarrow C = 0$

Substituting $C = 0$ in (2), we get $y = \frac{4x^3}{3(1+x^2)}$, which is the required solution.

Question 22

If $\hat{i} + \hat{j} + \hat{k}, 2\hat{i} + 5\hat{j}, 3\hat{i} + 2\hat{j} - 3\hat{k}$ and $\hat{i} - 6\hat{j} - \hat{k}$ respectively are the position vectors A, B, C and D, then find the angle between the straight lines AB and CD. Find whether $\overrightarrow{\text{AB}}$ and $\overrightarrow{\text{CD}}$ are collinear or not.

SOLUTION:

Given:

The position vector of A is $\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$.

The position vector of B is $2\hat{\boldsymbol{i}} + 5\hat{j}$.

Therefore, $\overrightarrow{AB} = (2-1)\hat{\boldsymbol{i}} + (5-1)\hat{j} + (0-1)\hat{k} = \hat{i} + 4\hat{j} - \hat{\boldsymbol{k}}$

The position vector of C is $3\hat{\boldsymbol{i}} + 2\hat{j} - 3\hat{k}$ and

The position vector of D is $\hat{\boldsymbol{i}} - 6\hat{j} - \hat{\boldsymbol{k}}$.

Therefore, $\overrightarrow{CD} = (1-3)\hat{i} + (-6-2)\hat{j} + (-1+3)\hat{\boldsymbol{k}} = -2\hat{\boldsymbol{i}} - 8\hat{j} + 2\hat{\boldsymbol{k}}$

Let θ be the angle between AB and CD, then

$\cos\theta = \frac{\overrightarrow{AB} \cdot \overrightarrow{CD}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{CD}\right|}$

$\Rightarrow \cos\theta = \frac{-2-32-2}{\sqrt{18}\sqrt{72}} = -1$

$\Rightarrow \theta = 180^\circ$

since angle between Line AB and CD is 180°, therefore $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear.

Question 23

Find the value of λ, so that the lines $\frac{1-x}{3}=\frac{7y-14}{\lambda}=\frac{z-3}{2}$ and $\frac{7-7x}{3\lambda}=\frac{y-5}{1}=\frac{6-z}{5}$ are at right angles. Also, find whether the lines are intersecting or not.

SOLUTION:

Given lines are $\frac{1-x}{3}=\frac{7y-14}{\lambda}=\frac{z-3}{2}$ and $\frac{7-7x}{3\lambda}=\frac{y-5}{1}=\frac{6-z}{5}$

Converting them into standard form, we have $\frac{x-1}{-3}=\frac{y-2}{\left(\lambda/7\right)}=\frac{z-3}{2}$ and

$\frac{x-1}{\left(-3\lambda/7\right)}=\frac{y-5}{1}=\frac{z-6}{-5}$

Corresponding d.r.'s are (($-3,\ \frac{\lambda}{7},\ 2$) and $\left(\frac{-3\lambda}{7},\ 1,\ -5\right)$

Since the angle between the lines is right angle so,

$$\cos 90^0=\left|\frac{(-3)\left(\frac{-3\lambda}{7}\right)+\left(\frac{\lambda}{7}\right)(1)+(2)(-5)}{\sqrt{(-3)^2+\left(\frac{\lambda}{7}\right)^2+2^2}\sqrt{\left(\frac{-3\lambda}{7}\right)^2+1^2+(-5)^2}}\right|$$

$$\Rightarrow 0=\left|\frac{\frac{9\lambda}{7}+\frac{\lambda}{7}-10}{\sqrt{\frac{\lambda^2}{49}+13}\sqrt{\frac{9\lambda^2}{49}+26}}\right|$$

Squaring and cross-multiplying

$\Rightarrow \left(\frac{10\lambda}{7}-10\right)^2=0$

$\Rightarrow \frac{10\lambda}{7}=10$

$\Rightarrow \lambda=7.$

Substituting the value of λ, the lines are $\frac{x-1}{-3}=\frac{y-2}{1}=\frac{z-3}{2}=a\left(\text{let}\right)$ and

$\frac{x-1}{-3}=\frac{y-5}{1}=\frac{z-6}{-5}=b$(let)

From first equation, $(x,\ y,\ z) = (-3a+1,\ a+2,\ 2a+3)$ and from second equation, $(x,\ y,\ z) = (-3b+1,\ b+5,\ -5b+6)$
Equating the corresponding values of coordinates, we have
$-3a+1 = -3b+1,\ a+2 = b+5 \text{ and } 2a+3 = -5b+6$
Or, $-3a+3b=0,\ a-b=3 \text{ and } 2a+5b=3$
Solving second and third equations of the above, we get $a=\frac{18}{7} \text{ and } b=\frac{-3}{7}$
Substituting these values of a and b in the first one
$-3\left(\frac{18}{7}\right)+3\left(\frac{-3}{7}\right)=-9$
Thus, it is clear that the first equation is not satisfied so the lines are not intersecting.

Question 24

If $A=\begin{bmatrix}1&1&1\\1&0&2\\3&1&1\end{bmatrix}$, find A^{-1}. Hence, solve the system of equations $x+y+z=6$, $x+2z=7$, $3x+y+z=12$.

OR

Find the inverse of the following matrix using elementary operations.

$$A=\begin{bmatrix}1&2&-2\\-1&3&0\\0&-2&1\end{bmatrix}$$

SOLUTION:

Given Matrix $A=\begin{bmatrix}1&1&1\\1&0&2\\3&1&1\end{bmatrix}$
To find A^{-1}, we need cofactors of each element of matrix A.

$$\text{cofactor of } a_{11}=(-1)^{1+1}\begin{vmatrix}0&2\\1&1\end{vmatrix}=-2$$

$$\text{cofactor of } a_{12}=(-1)^{1+2}\begin{vmatrix}1&2\\3&1\end{vmatrix}=-(1-6)=5$$

$$\text{cofactor of } a_{13}=(-1)^{1+3}\begin{vmatrix}1&0\\3&1\end{vmatrix}=1$$

$$\text{cofactor of } a_{21} = (-1)^{2+1}\begin{vmatrix}1 & 1\\ 1 & 1\end{vmatrix} = 0$$

$$\text{cofactor of } a_{22} = (-1)^{2+2}\begin{vmatrix}1 & 1\\ 3 & 1\end{vmatrix} = (1-3) = -2$$

$$\text{cofactor of } a_{23} = (-1)^{2+3}\begin{vmatrix}1 & 1\\ 3 & 1\end{vmatrix} = -(1-3) = 2$$

$$\text{cofactor of } a_{31} = (-1)^{3+1}\begin{vmatrix}1 & 1\\ 0 & 2\end{vmatrix} = 2$$

$$\text{cofactor of } a_{32} = (-1)^{3+2}\begin{vmatrix}1 & 1\\ 1 & 2\end{vmatrix} = -(2-1) = -1$$

$$\text{cofactor of } a_{33} = (-1)^{3+3}\begin{vmatrix}1 & 1\\ 1 & 0\end{vmatrix} = -1$$

$$\text{So cofactor of matrix of } A = \begin{bmatrix}-2 & 5 & 1\\ 0 & -2 & 2\\ 2 & -1 & -1\end{bmatrix}$$

$\because$ the trnspose of cofactor matrix A is adj (A)

$$\text{So adj } (A) = \begin{bmatrix}-2 & 0 & 2\\ 5 & -2 & -1\\ 1 & 2 & -1\end{bmatrix}$$

$|A| = 1(0-2) - 1(1-6) + 1(1-0)$
$= -2 + 5 + 1$
$= 4$

$\& \ A^{-1} = \frac{1}{|A|}$ adj (A)

$$\text{so, } A^{-1} = \frac{1}{4}\begin{bmatrix}-2 & 0 & 2\\ 5 & -2 & -1\\ 1 & 2 & -1\end{bmatrix}$$

Now, the given system of eq^n is

$x + y + z = 6$
$x + 2z = 7$
$3x + y + z = 12$
Writing the above equation in matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$$

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$$

$$or, \quad \frac{1}{4} \begin{bmatrix} -12 + 0 + 24 \\ 30 - 14 - 12 \\ 6 + 14 - 12 \end{bmatrix}$$

$$or, \quad \frac{1}{4} \begin{bmatrix} 12 \\ 4 \\ 8 \end{bmatrix}$$

$$or, \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$i.e\ x = 3,\ y = 1,\ z = 2$

OR

We know that

A = I A

$$\text{or, } \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \text{ [Applying } R_2 \rightarrow R_2 + R_1]$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} A \text{ [Applying } R_2 \rightarrow R_2 + 2R_3]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -4 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} A \text{ [Applying } R_1 \rightarrow R_1 + (-2)R_2, R_3 \rightarrow R_3 + 2R_2]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} A \text{ [Applying } R_1 \rightarrow R_1 + 2R_3]$$

$$\text{Hence, } A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

Question 25

A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3. If building of tank costs ₹ 70 per square metre for the base and ₹ 45 per square metre for the sides, what is the cost of least expensive tank?

SOLUTION:

Let *l*, *b* and *h* be the length, breadth and height of the tank, respectively.

Height, $h = 2$ m

Volume of the tank = 8 m^3

Volume of the tank = $l \times b \times h$

$\therefore l \times b \times 2 = 8$

$$\Rightarrow lb = 4$$
$$\Rightarrow b = \frac{4}{l}$$

Area of the base = $lb = 4\ m^2$

Area of the 4 walls, $A = 2h(l + b)$

$$\therefore A = 4\left(l + \frac{4}{l}\right)$$
$$\Rightarrow \frac{dA}{dl} = 4\left(1 - \frac{4}{l^2}\right)$$

For maximum or minimum values of A, we must have

$$\frac{dA}{dl} = 0$$
$$\Rightarrow 4\left(1 - \frac{4}{l^2}\right) = 0$$
$$\Rightarrow l = \pm 2$$

However, the length cannot be negative.

Thus,
$l = 2$ m

$$\therefore b = \frac{4}{2} = 2 \text{ m}$$

Now,

$$\frac{d^2A}{dl^2} = \frac{32}{l^3}$$

At $l = 2$:

$$\frac{d^2A}{dl^2} = \frac{32}{8} = 4 > 0$$

Thus, the area is the minimum when $l = 2$ m

We have
$l = b = h = 2$ m

$\therefore$ Cost of building the base = Rs 70 × (lb) = Rs 70 × 4 = Rs 280

Cost of building the walls = Rs $2h(l + b)$ × 45 = Rs 90 (2) (2 + 2)= Rs 8 (90) = Rs 720

Total cost = Rs (280 + 720) = Rs 1000

Hence, the total cost of the tank will be Rs 1000.

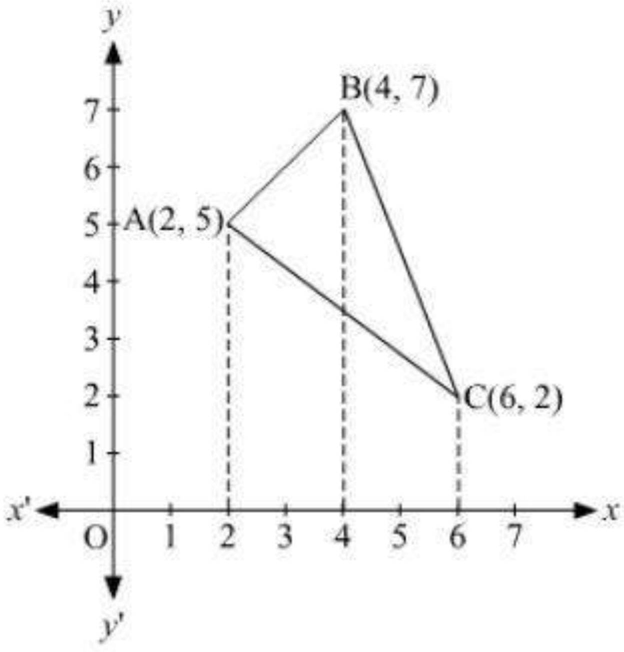
Question 26

Using integration, find the area of triangle ABC, whose vertices are A(2, 5), B(4, 7) and C(6, 2).

OR

Find the area of the region lying above x-axis and included between the circle $x^2 + y^2 = 8x$ and inside of the parabola $y^2 = 4x$.

SOLUTION:



Vertices of the given triangle are A(2,5), B(4,7) and C(6,2).
Equation of AB

$y-5=\frac{7-5}{4-2}(x-2)$

$\Rightarrow y-5=x-2$

$\Rightarrow y=x+3$

Let's say $y_1=x+3$
Equation of BC :

$y-7=\frac{2-7}{6-4}(x-4)$

$\Rightarrow y=\frac{-5}{2}(x-4)+7=\frac{-5}{2}x+17$

Let's say $y_2=-\frac{5}{2}x+17$
Equation of AC:

$y-5=\frac{2-5}{6-2}(x-2)$

$\Rightarrow y=\frac{-3}{4}(x-2)+5=\frac{-3}{4}x+\frac{13}{2}$

Let's say $y_3=\frac{-3}{4}x+\frac{13}{2}$

$\operatorname{ar}(\Delta \mathrm{ABC})=\int_2^4 y_1\,\mathrm{d}x+\int_4^6 y_2\,\mathrm{d}x-\int_2^6 y_3\,\mathrm{d}x$

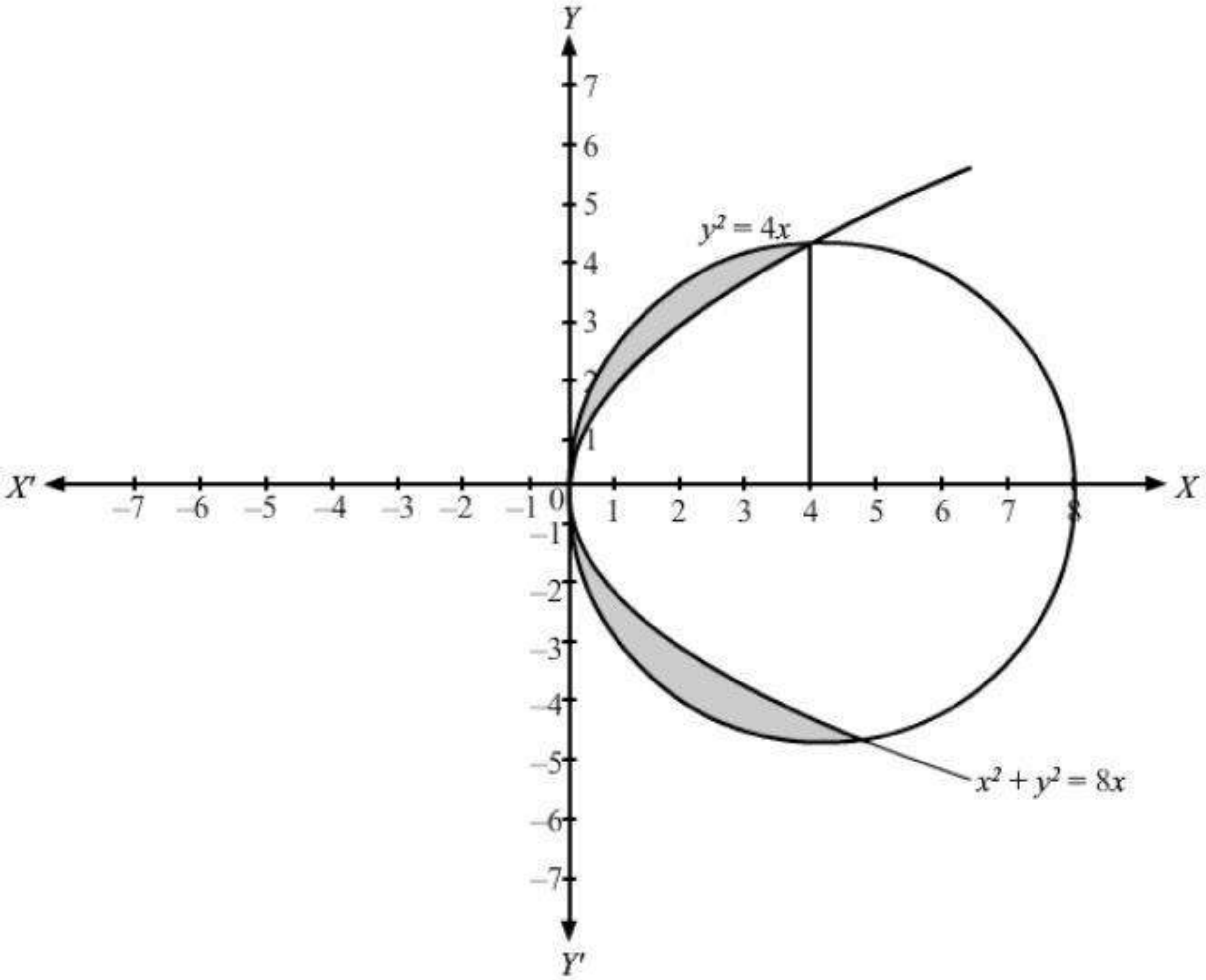
$=\int_2^4(x+3)\,\mathrm{d}x+\int_4^6\left(\frac{-5}{2}x+17\right)\mathrm{d}x-\int_2^6\left(\frac{-3}{4}x+\frac{13}{2}\right)\mathrm{d}x$

$=\left[\frac{x^2}{2}+3x\right]_2^4+\left[\frac{-5x^2}{4}+17x\right]_4^6-\left[\frac{-3x^2}{8}+\frac{13x}{2}\right]_2^6$

$=\left[\frac{16}{2}+12-\frac{4}{2}-6\right]+\left[\frac{-180}{4}+102+\frac{80}{4}-68\right]-\left[\frac{-108}{8}+\frac{78}{2}+\frac{12}{8}-\frac{26}{2}\right]$

= 12 + 9 + 14
= 35 sq. units.

OR



The given equations are $x^2 + y^2 = 8x \quad \cdots (1)$ and $y^2 = 4x \quad \cdots (2)$
Clearly the equation $x^2 + y^2 = 8x$ is a circle with centre $(4, 0)$ and has a radius 4.Also $y^2 = 4x$ is a parabola with vertex at origin and the axis along the x-axis opening in the positive direction .
To find the intersecting points of the curves ,we solve both the equation.
$\therefore x^2 + 4x = 8x$
$\Rightarrow x^2 - 4x = 0$
$\Rightarrow x(x - 4) = 0$
$\Rightarrow x = 0 \text{ and } x = 4$
When $x = 0, \ y = 0$
When $x = 4, \ y = \pm 4$

To approximate the area of the shaded region the length $= |y_2 - y_1|$ and the width = *dx*

$$A = \int_0^4 |y_2 - y_1| \, \mathrm{d}x$$
$$= \int_0^4 (y_2 - y_1) \, \mathrm{d}x \quad [\because y_2 > y_1 \quad \therefore |y_2 - y_1| = y_2 - y_1]$$
$$= \int_0^4 \left[\sqrt{\left(16 - (x-4)^2\right)} - \sqrt{4x}\right] dx \quad \left\{\therefore y_2 = \sqrt{16 - (x-4)^2} \text{ and } y_1 = 2\sqrt{x}\right\}$$
$$= \int_0^4 \sqrt{16 - (x-4)^2} \, \mathrm{d}x - \int_0^4 \sqrt{4x} \, \mathrm{d}x$$
$$= \left[\frac{(x-4)}{2}\sqrt{16 - (x-4)^2} + \frac{16}{2}\sin^{-1}\left(\frac{x-4}{4}\right)\right]_0^4 - \left[\frac{4x^{\frac{3}{2}}}{3}\right]_0^4$$
$$= \left[0 + 0 - 0 - 8\sin^{-1}\left(\frac{-4}{4}\right)\right] - \frac{4}{3} \times 4^{\frac{3}{2}}$$
$$= \frac{8\pi}{2} - \frac{32}{3}$$
$$= 4\pi - \frac{32}{3}$$

Hence the required area is $4\pi - \frac{32}{3}$ square units.

Question 27

Find the vector and Cartesian equations of the plane passing through the points (2, 2 − 1), (3, 4, 2) and (7, 0, 6). Also find the vector equation of a plane passing through (4, 3, 1) and parallel to the plane obtained above.

OR

Find the vector equation of the plane that contains the lines $\vec{r} = \left(\hat{i} + \hat{j}\right) + \lambda\left(\hat{i} + 2\hat{j} - \hat{k}\right)$ and the point (−1, 3, −4). Also, find the length of the perpendicular drawn from the point (2, 1, 4) to the plane thus obtained.

SOLUTION:

step 1

The given points are A(2, 2, −1), B(3, 4, 2) and C(7, 0, 6)

Let $\vec{a} = 2\hat{i} + 2\hat{j} - \hat{k}$

$\vec{b} = 3\hat{i} + 4\hat{j} + 2\hat{k}$

$\vec{c} = 7\hat{i} + 6\hat{k}$

Hence the vector equation of the plane passing through the points

$$\left(\vec{r} - \vec{a}\right) \cdot \left(\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{AC}}\right) = 0$$

$$=\left(\vec{r}-\vec{a}\right)\cdot\left(\left(\vec{b}-\vec{a}\right)\times\left(\vec{c}-\vec{a}\right)\right)=0$$

Now

$$\vec{b}-\vec{a}=\left(3\hat{i}+4\hat{j}+2\hat{k}\right)-\left(2\hat{i}+2\hat{j}-\hat{k}\right)$$

$$\Rightarrow\hat{i}+2\hat{j}+3\hat{k}$$

$$\vec{c}-\vec{a}=\left(7\hat{i}+6\hat{k}\right)-\left(2\hat{i}+2\hat{j}-\hat{k}\right)$$

$$=5\hat{i}-2\hat{j}+7\hat{k}$$

So the required vector equation of plane is

$$\left[\vec{r}-\left(2\hat{i}+2\hat{j}-\hat{k}\right)\right]\cdot\left[\left(\hat{i}+2\hat{j}+3\hat{k}\right)\times\left(5\hat{i}-2\hat{j}+7\hat{k}\right)\right]=0$$

Step 2

$$\left(\vec{b}-\vec{a}\right)\times\left(\vec{c}-\vec{a}\right)=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&2&3\\5&-2&7\end{vmatrix}$$

$$=\hat{i}\left(14+6\right)-\hat{j}\left(7-15\right)+\hat{k}\left(-2-10\right)$$

$$=20\hat{i}+8\hat{j}-12\hat{k}$$

$$\Rightarrow\left(\vec{r}-\left(2\hat{i}+2\hat{j}-\hat{k}\right)\right)\cdot\left(20\hat{i}+8\hat{j}-12\hat{k}\right)=0$$

$$\left(\vec{r}-\left(2\hat{i}+2\hat{j}-\hat{k}\right)\right)\cdot\left(5\hat{i}+2\hat{j}-3\hat{k}\right)=0$$

$$\vec{r}\cdot\left(5\hat{i}+2\hat{j}-3\hat{k}\right)=\left(2\hat{i}+2\hat{j}-\hat{k}\right)\cdot\left(5\hat{i}+2\hat{j}-3\hat{k}\right)$$

$$\vec{r}\cdot\left(5\hat{i}+2\hat{j}-3\hat{k}\right)=10+4+3$$

$$\vec{r}\cdot\left(5\hat{i}+2\hat{j}-3\hat{k}\right)=17$$

This is the required vector equation of the plane

Step 3

The Cartesian Equation of the plane passing through the three points is given as below-

$5x + 2y - 3z - 17 = 0$

This is required cartesian equation of the plane.

The equation of plane parallel to $5x + 2y - 3z - 17 = 0$ will be $5x + 2y - 3z + \lambda = 0$

$\therefore$ it passes through (4, 3, 1).

So, $5 \times 4 + 2 \times 3 - 3 \times 1 + \lambda = 0$

$20 + 6 - 3 + \lambda = 0$

So, $\lambda = -23$

so the equation of the plane will be

$5x + 2y - 3z - 23 = 0$

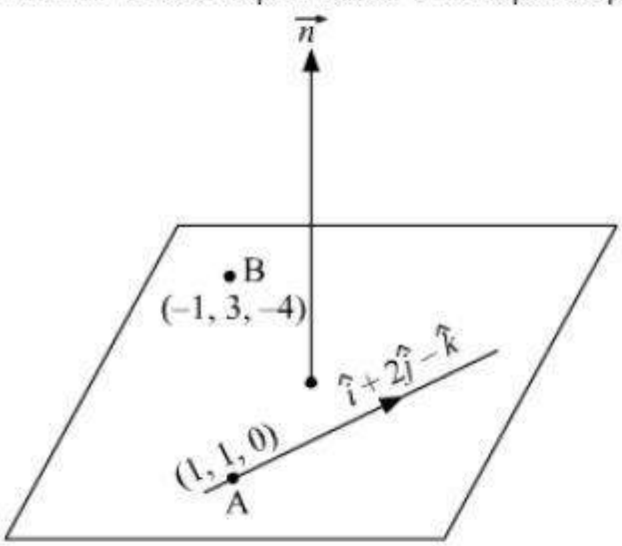
$5x + 2y - 3z = 23$

so the vector form of the equation of plane will be

$\vec{r} \cdot \left(5\hat{i} + 2\hat{j} - 3\hat{k}\right) = 23$

OR

Let the vector equation of the required plane be $\vec{r} \cdot \vec{n} = d$



The plane contains the line $\vec{r} = \hat{i} + \hat{j} + \lambda\left(\hat{i} + 2\hat{j} - \hat{k}\right)$

Since the plane passes through point A and B. So $\overrightarrow{n}$ will be parallel to vector

$$\overrightarrow{AB} \times \left(\hat{i} + 2\hat{j} - \hat{k}\right)$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \left(-\hat{i} + 3\hat{j} - 4\hat{k}\right) - \left(\hat{i} + \hat{j}\right)$$

$$= -2\hat{i} + 2\hat{j} - 4\hat{k}$$

$$\overrightarrow{AB} \times \left(\hat{i} + 2\hat{j} - \hat{k}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ -2 & 2 & -4 \end{vmatrix}$$

$$= \hat{i}\,(-8 + 2) - \hat{j}\,(-4 - 2) + \hat{k}\,(2 + 4)$$

$$= -6\hat{i} + 6\hat{j} + 6\hat{k}$$

which is a normal vector to the plane.

So the equation of plane will be $\overrightarrow{r} \cdot \left(-6\hat{i} + 6\hat{j} + 6\hat{k}\right) = d$

$\therefore$ it passes through (1, 1, 0) so $\left(\hat{i} + \hat{j}\right) \cdot \left(-6\hat{i} + 6\hat{j} + 6\hat{k}\right) = d$ or, $d = 0$

equation of plane is $\overrightarrow{r} \cdot \left(-6\hat{i} + 6\hat{j} + 6\hat{k}\right) = 0$

$$\overrightarrow{r}\left(\hat{i} - \hat{j} - \hat{k}\right) = 0$$

in Cartesian plane,

$$\left(x\hat{i} + y\hat{j} + z\hat{k}\right) \cdot \left(\hat{i} - \hat{j} - \hat{k}\right) = 0$$

$x - y - z = 0$

So, the perpendicular distance of the plane from the point (2, 1, 4) is

$$= \left|\frac{2-1-4}{\sqrt{1^2+(-1)^2+(-1)^2}}\right| = \left|\frac{-3}{\sqrt{3}}\right| = \sqrt{3}\,\text{unit}.$$

Question 28

A manufacture has three machine operators A, B and C. The first operator A produces 1% of defective items, whereas the other two operators B and C produces 5% and 7% defective items respectively. A is on the job for 50% of the time, B on the job 30% of the time and C on the job for 20% of the time. All the items are put into one stockpile and then one item is chosen at random from this and is found to be defective. What is the probability that it was produced by A?

SOLUTION:

Let E_1, E_2 and E_3 be the event that machine is operated by A, B, and C respectively.

Let A be the event of producing defective items.

$$\therefore\ P(E_1) = 50\% = \frac{1}{2}$$
$$P(E_2) = 30\% = \frac{3}{10}$$
$$P(E_3) = 20\% = \frac{1}{5}$$

Now,

$$P(A/E_1) = 1\% = \frac{1}{100}$$
$$P(A/E_2) = 5\% = \frac{5}{100}$$
$$P(A/E_2) = 7\% = \frac{7}{100}$$

Using Bayes' theorem, we get

$$\text{Required probability } = P(E_1/A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + + P(E_2)P(A/E_2)}$$
$$= \frac{\frac{1}{2} \times \frac{1}{100}}{\frac{1}{2} \times \frac{1}{100} + \frac{3}{10} \times \frac{5}{100} + \frac{1}{5} \times \frac{7}{100}}$$
$$= \frac{5}{34}$$

Question 29

A manufacturer has employed 5 skilled men and 10 semi-skilled men and makes two models A and B of an article. The making of one item of model A requires 2 hours work by a skilled man and 2 hours work by a semi-skilled man. One item of model B requires 1 hour by a skilled man and 3 hours by a semi-skilled man. No man is expected to work more than 8 hours per day. The manufacturer's profit on an item of model A is ₹ 15 and on an item of model B is ₹ 10. How many of items of each model should be made per day in order to maximize daily profit? Formulate the above LPP and solve it graphically and find the maximum profit.

SOLUTION:

Let x articles of model A and y articles of model B be made.

Number of articles cannot be negative.

Therefore, $x, y \geq 0$

According to the question, the making of a model A requires 2 hrs. work by a skilled man and the model B requires 1 hr by a skilled man

$2x + y \leq 40$

The making of a model A requires 2 hrs. work by a semi-skilled man model B requires 3 hrs. work by a semi-skilled man.

$$2x + 3y \leq 80$$

Total profit = Z = $15x + 10y$ which is to be maximised

Thus, the mathematical formulation of the given linear programmimg problem is

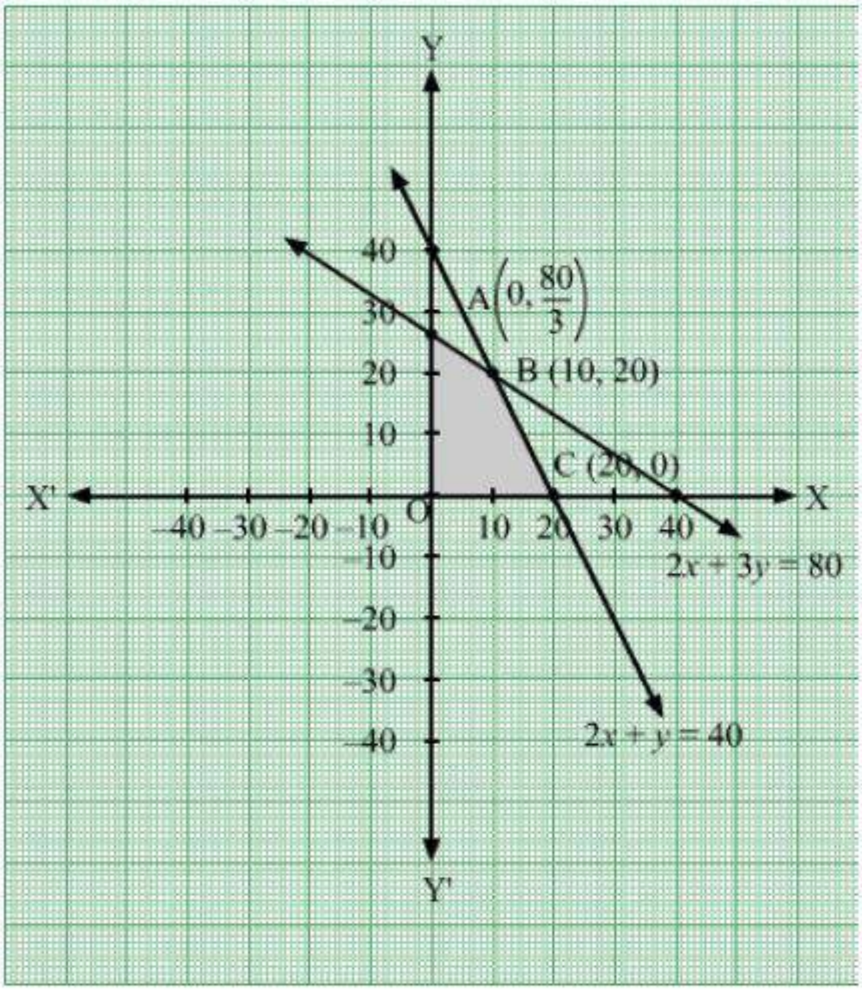
Max Z = $15x + 10y$

subject to

$$2x + y \leq 40$$
$$2x + 3y \leq 80$$

$$x \geq 0$$
$$y \geq 0$$

The feasible region determined by the system of constraints is



The corner points are A$(0, \frac{80}{3})$, B(10, 20), C(20, 0)

The values of Z at these corner points are as follows

Corner point	Z= $15x+10y$
A	$\frac{800}{3}$
B	350
C	300

The maximum value of Z is 300 which is attained at C(20, 0)

Thus, the maximum profit is Rs 300 obtained when 10 units of deluxe model and 20 unit of ordinary model is produced.